Last time:

$$T: D^* \subset \mathbb{R}^2 \longrightarrow D \subset \mathbb{R}^2$$

area of $T(R) \approx |\det dT(u,v)|\, \Delta u \Delta v$

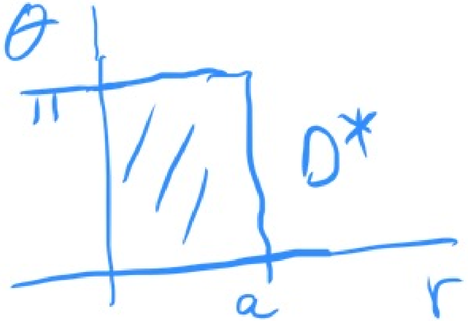
applications:

Theorem 1 If $T: D^* \longrightarrow D$ is 1-1 and onto

$$\Rightarrow \text{area } T(D^*) = \iint_{D^*} |dT| \, du\, dv$$

$$|dT| = |\det dT|$$

Example: Calculate area of half-disk of radius a using polar coordinates



given via polar coordinates

$$0 \le r \le a$$
$$0 \le \theta \le \pi$$

$$T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$dT = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\Rightarrow |dT| = \left| \det \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \right|$$

$$= | r\cos^2\theta - (-r\sin\theta)\sin\theta |$$

$$= | r(\cos^2\theta + \sin^2\theta) |$$

$$= r$$

$\Rightarrow$ area of half disk $= \iint_D 1 \cdot dx\,dy$

$$= \iint_{D^*} |dT| \, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{a} r\, dr\, d\theta = \int_0^{\pi} \frac{r^2}{2}\Big|_0^a d\theta = \int_0^{\pi} \frac{a^2}{2} d\theta = \boxed{\frac{1}{2}\pi a^2}$$

Change of Variable Theorem

$T : D^* \to D$ 1-1 and onto

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(\underbrace{x(u,v),\, y(u,v)}_{=T(u,v)})\,|dT|\,du\,dv$$

where $T(u,v) = (x(u,v),\, y(u,v))$

Example: Calculate $\iint_D \log(x^2+y^2)\,dx\,dy$

where $D =$ 

use polar coordinates:

$1 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$ $\Big\}$ describes our D^*

as before $|dT| = r$

$$\iint_D \exp(x^2+y^2)\,dx\,dy = \iint_{D^*} \exp(r^2) \cdot r\,dr\,d\theta$$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2$$

$$= \int_0^{\pi/2}\int_1^2 e^{r^2} \cdot \underset{|dT|}{r}\,dr\,d\theta$$

used $\frac{d}{dr}(e^{r^2}) = e^{r^2}\cdot 2r$

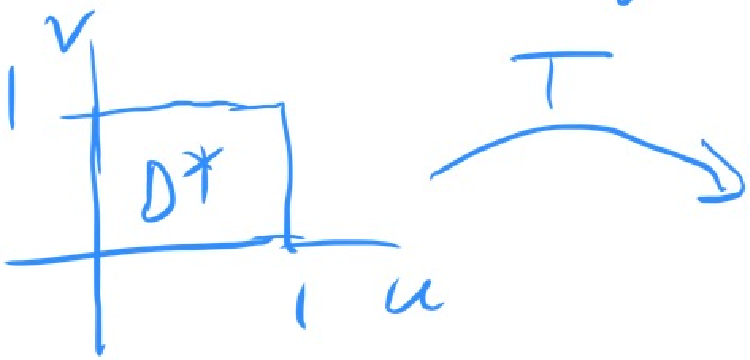
$$= \int_0^{\pi/2} \frac{1}{2}e^{r^2}\Big|_{r=1}^{r=2} d\theta = \int_0^{\pi/2} \frac{1}{2}(e^4-e^1)\,d\theta$$

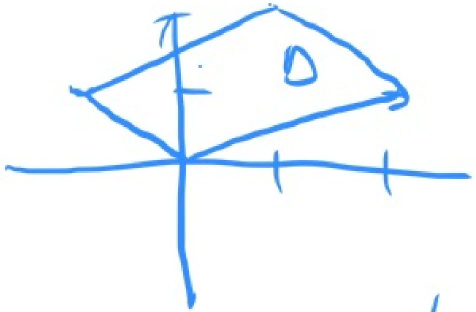
$$= \frac{1}{2}(e^4-e)\frac{\pi}{2}$$

Example 2

Let D be the parallelogram spanned by the vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Calculate the integral $\iint_D x^2 y \, dx\,dy$





$$T(u,v) = u\begin{pmatrix} 2 \\ 1 \end{pmatrix} + v\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2u - v \\ u + v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$D = \left\{ u\begin{pmatrix} 2 \\ 1 \end{pmatrix} + v\begin{pmatrix} -1 \\ 1 \end{pmatrix} \;\middle|\; \begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 1 \end{matrix} \right\}$$

$$x = 2u - v \qquad\qquad dT = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

$$y = u + v$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow |dT| = |2\cdot 1 - (-1)\cdot 1| = 3$$

$$\Rightarrow \iint_D x^2 y \, dx\,dy = \int_0^1\int_0^1 \underbrace{(2u-v)^2}_{x^2}\underbrace{(u+v)}_{y} \cdot \underbrace{3}_{|dT|} \, du\,dv$$

$$\iint_{D^*}$$

$$D^* = [0,1]\times[0,1] \qquad = \int_0^1\int_0^1 3(4u^2 - 4uv + v^2)(u+v)\,du\,dv$$

$$= \;\ldots\ldots\; 9/4 \quad \text{(check!)}$$

Change of variable formula in 3-dimension

$$T: W^* \subset \mathbb{R}^3 \longrightarrow W \subset \mathbb{R}^3$$

1-1 and onto.

$$\boxed{\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(T(u,v,w)) \cdot |dT|\,du\,dv\,dw}$$

Example: spherical coordinates